Profile, Plan and Streamline Curvature: A Simple Derivation and Applications

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***Abstract*—Profile and plan curvature are standard tools in geomorphometry. Streamline curvature has not been as widely used but is a natural counterpart to plan curvature. Expressions for all three types of curvature can be easily derived using the concept of a *directional derivative*. Various types of curvature can sometimes be used to make general statements about the solutions to a PDE (partial differential equation). Examples are given to show how nonlinear, first and second-order PDEs can be manipulated with simple algebra to get expressions regarding their solutions in terms of curvatures. These expressions are powerful in that they allow us to use our intuitive, geometric understanding of curvature concepts to better understand nonlinear PDEs for which it is often difficult to obtain analytical or numerical solutions.**

I. Background

A. Slope and Aspect

Let $f(x,y)$ denote a surface and $\nabla f = (f_x, f_y, 0) = f_x\hat{i} + f_y\hat{j}$ denote the gradient of this surface (a 2D vector field). Note that the subscripts x and y indication partial derivatives. Recall that ∇f and $-\nabla f$ point in the steepest uphill and downhill directions, respectively, but ∇f has no z-component. The *slope* at each point is defined as the magnitude of the gradient vector and is a scalar field

$$S(x,y) = |\nabla f| = \sqrt{f_x^2 + f_y^2}. \tag{1}$$

Note that $S \geq 0$. The *aspect* at each point is defined as

$$\phi(x,y) = \arctan\left(\frac{f_y}{f_x}\right). \tag{2}$$

Aspect is the angle that the (2D) gradient vector, ∇f, makes with a fixed, but arbitrary x-axis. Note that most programming languages provide a two-parameter $\arctan$ function that returns a value in the correct quadrant. Since ∇f points uphill, the flow direction angle is $\phi(x,y)+\pi$. Let $\nabla^{\perp} f = (f_y, -f_x)$ denote a vector field that is everywhere perpendicular to the gradient of f (to the right of ∇f and to the left of $-\nabla f$). Note that $\left(\nabla^{\perp} f \cdot \nabla f\right) = 0$, as required to be perpendicular.

B. Directional Derivatives

Recall that a *directional derivative* can be used to compute the rate at which any given scalar field, $F(x,y)$, is changing as we move in the direction of some unit vector, $\hat{n}$. That is,

$$D_{\hat{n}}(F) = \nabla F \cdot \hat{n}. \tag{3}$$

This expression can be used to define several different kinds of curvature as will be seen in the following section. Note that ∇F is a 2D vector field and $\hat{n}$ can also be a 2D vector field, such as $\hat{n} = (-\nabla f/S)$.

II. Derivations of Curvature

A. Profile Curvature

Profile curvature is the rate at which surface slope, S, changes as we move in the direction of the unit vector $(-\nabla f/\mathrm{S})$, (i.e. following a streamline downstream). It can be expressed in coordinate-free form as:

$$\kappa_p = -S^{-1}\left(\nabla S \cdot \nabla f\right). \tag{4}$$

To get an expression in Cartesian coordinates, we start with the definition of slope, $S(x,y)$, and the fact that $\nabla S =$

(S_x, S_y). Taking derivatives of S and simplifying, we get

$$S\,S_x = (f_x\,f_{xx} + f_y\,f_{xy}) \tag{5}$$
$$S\,S_y = (f_x\,f_{xy} + f_y\,f_{yy})\,. \tag{6}$$

Using these, we get

$$\kappa_p = -S^{-2}\left(f_x^2\,f_{xx} + 2\,f_x\,f_y\,f_{xy} + f_y^2\,f_{yy}\right). \tag{7}$$

A longitudinal profile for a streamline passing through (x, y) is locally *concave up* if $\kappa_p(x, y) < 0$, and convex (or concave down) otherwise. As stated by [3], the convention in geomorphometry is to define curvatures as negative for concavities and positive for convexities. An extra factor of $\left(1 + S^2\right)^{3/2}$ appears in the denominator if we define curvature based on differential movements along the 3D streamline curve (that lies on the surface) instead of along the 2D streamline curve (that lies in the xy plane).

B. Plan Curvature

Plan or contour curvature is the rate at which flow direction (given by $\phi + \pi$) changes as we move in the direction of $(\nabla^{\perp} f/S)$, (i.e. following a contour line). It can be expressed in coordinate-free form as:

$$\kappa_c = S^{-1}\left(\nabla\phi \cdot \nabla^{\perp} f\right). \tag{8}$$

To get an expression in Cartesian coordinates, we start with (2) and the fact that $\nabla\phi = (\phi_x, \phi_y)$. Recall that $\frac{d}{dx}\left[\arctan(x)\right] = 1/(1 + x^2)$ in all quadrants. Taking derivatives of ϕ and simplifying, gives

$$S^2\,\phi_x = (f_x\,f_{xy} - f_y\,f_{xx}) \tag{9}$$
$$S^2\,\phi_y = (f_x\,f_{yy} - f_y\,f_{xy}) \tag{10}$$

which finally leads to

$$\kappa_c = -S^{-3}\left(f_y^2\,f_{xx} - 2\,f_x\,f_y\,f_{xy} + f_x^2\,f_{yy}\right). \tag{11}$$

Note that $\kappa_c(x, y)$ measures the curvature of the contour line (a curve in the xy plane) that goes through the point (x, y). Plan curvature is negative for valleys and positive for ridges. See the Applications section. Plan curvature is closely related to the *tangential curvature*, given by

$$\kappa_t = \frac{f_y^2\,f_{xx} - 2\,f_x\,f_y\,f_{xy} + f_x^2\,f_{yy}}{-S^2\sqrt{1 + S^2}}. \tag{12}$$

The key difference is that κ_t is a *normal curvature*. The extra function of S in the denominator is due to the angle between the surface normal and the xy plane. That is, $\kappa_t(x, y)$ measures the curvature of the 3D curve formed by the intersection of the surface $f(x, y)$ and the plane that contains both the surface normal vector at (x, y) and $\nabla^{\perp} f$. Expressions for plan curvature should not have extra functions of S in the denominator.

C. Streamline Curvature

By analogy with how plan and profile curvature are defined, we can define *streamline curvature* as the rate at which the flow direction $(\phi + \pi)$ changes as we move in the direction of $(-\nabla f/S)$, (i.e. following a streamline). It is the inverse of a streamline's local *radius of curvature*, which measures how tightly the streamline bends. It can be expressed in coordinate-free form as:

$$\kappa_s = -S^{-1}\left(\nabla\phi \cdot \nabla f\right). \tag{13}$$

To express this in Cartesian coordinates we can use (9) and (10) again to get

$$\kappa_s = -S^{-3}\left[f_x\,f_y\,\left(f_{xx} - f_{yy}\right) + \left(f_y^2 - f_x^2\right)\,f_{xy}\right]. \tag{14}$$

According to [10], this type of curvature was first considered by [8] but neither this nor plan curvature belong to the "complete system of curvatures" discussed by [9]. The sign distinguishes right from left, but typically only the magnitude (absolute value) would be of interest.

III. Applications of Curvature

A. A Descending Ridge or Valley

A parabolic sheet surface atop an inclined plane, given by $f(x, y) = a\,x^2 + b\,y$, provides a simple model of a descending ridge or valley. This surface has $S(x, y) = \sqrt{4\,a^2\,x^2 + b^2}$ and

$$\kappa_c(x, y) = -2\,a\,b^2\,S^{-3} \tag{15}$$
$$\kappa_p(x, y) = -8\,a^3\,x^2\,S^{-2} \tag{16}$$
$$\kappa_s(x, y) = -4\,a^2\,b\,x\,S^{-3}. \tag{17}$$

For $a > 0$, it looks like a parabolic valley (concave up) and has $\kappa_c < 0$ everywhere. For $a < 0$, it looks like a parabolic ridge (convex) and has $\kappa_c > 0$ everywhere. The centerline of the ridge or valley is given by the line $x = 0$ and is a line of constant slope. For this example, it is possible to obtain expressions for the contour and streamline curves as functions of x

$$y_c(x) = (u/b) - (a/b)x^2 \tag{18}$$

$$y_s(x) = v + b\,\ln(x^2)/(4a) \tag{19}$$

where different values of u and v give different contours and streamlines. Applying the standard formula for curvature of a plane curve reproduces $\kappa_c(x,y)$ and $\kappa_s(x,y)$ as given above and serves as a simple check.

B. The Equation of Geometrical Optics

There is an interesting class of surfaces such that the slope, $S(x,y)$, at every point on the surface is the same; see [1], [5]. These surfaces satisfy the nonlinear, first-order PDE

$$\left(f_x^2 + f_y^2\right) = S^2 = c > 0. \tag{20}$$

This is known as the *equation of geometrical optics* because its streamlines describe the straight paths of light rays in optical equipment (e.g. lenses and mirrors). Planes and cones are simple examples of such surfaces. Since S is constant everywhere, we expect that all such surfaces should have the profile curvature, $\kappa_p = 0$ everywhere. To prove that this is the case, we take the x-derivative of (20) and the y-derivative of (20) to generate two new equations

$$f_x\,f_{xx} + f_y\,f_{xy} = 0 \tag{21}$$

$$f_y\,f_{yy} + f_x\,f_{xy} = 0. \tag{22}$$

Multiplying (21) by f_x and (22) by f_y and then adding gives

$$f_x^2\,f_{xx} + 2\,f_x\,f_y\,f_{xy} + f_y^2\,f_{yy} = 0. \tag{23}$$

This shows that $\kappa_p = 0$ everywhere; see (7). But if the streamlines of such surfaces are always straight lines, then we should also have $\kappa_s = 0$ at *every* point on *all* such surfaces. This can be proven in a similar manner, by multiplying (21) by f_y and (22) by f_x and then subtracting to get

$$f_x\,f_y\,(f_{xx} - f_{yy}) + \left(f_y^2 - f_x^2\right) f_{xy} = 0. \tag{24}$$

This shows that $\kappa_s = 0$ everywhere; see (14).

A general, parametric solution to (20) (with c=1) can be found using the method developed by [5] and is given by

$$x(u,v) = u\cos\left[\phi(v)\right] - \textstyle\int C(v)\sin\left[\phi(v)\right]\,dv \tag{25}$$

$$y(u,v) = u\sin\left[\phi(v)\right] + \textstyle\int C(v)\cos\left[\phi(v)\right]\,dv \tag{26}$$

$$z(u,v) = u \tag{27}$$

where $\phi(v)$ and $C(v)$ are arbitrary functions of v, and $\phi(v)$ is the aspect angle for a constant-v streamline. Note that $x(u,v)$ and $y(u,v)$ describe an orthogonal, curvilinear coordinate system where constant-u curves are contours and constant-v curves are streamlines. Orthogonality is easy to check: $x_u x_v + y_u y_v = 0$. A plane, $f(x,y) = a\,x + b\,y + c$, is obtained for $\phi(v) = \phi_0$. A cone, $f(x,y) = \sqrt{x^2+y^2} + c$, is obtained for $\phi(v) = v$ and $C(v) = -c$. Fig. 1 shows the surface obtained when $C(v) = 1$ and $\phi(v) = -a\,\cos(v)$ The $u = 0$ contour line is the *sine-generated curve* model for meandering streams introduced by [2], with $\phi'(v) = a\,\sin(v)$, $v =$ arclength, $a = 1.8$ and $u \in [-0.3, 0.3]$.

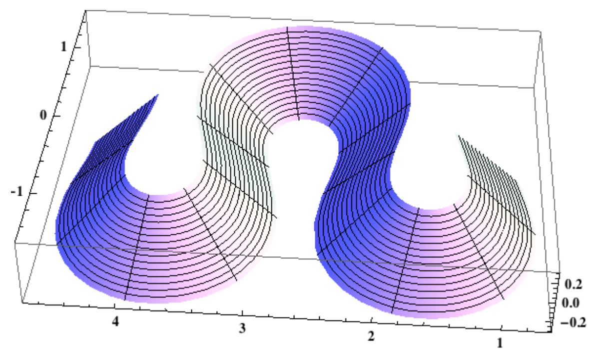

Fig. 1. A meander sheet surface with a constant slope of 1 everywhere.

C. Laplace's Equation

Laplace's equation is a linear, elliptic, second-order PDE that is used as a model in many different contexts from electrostatics to aerodynamics. It is given by

$$\nabla^2 f = (f_{xx} + f_{yy}) = 0. \tag{28}$$

Using our previous expressions for plan (7) and profile curvature (11) we can rewrite Laplace's equation as

$$\kappa_p + S\,\kappa_c = 0. \tag{29}$$

Since we always have $S \geq 0$, this shows that at any given point (x, y), the plan and profile curvatures for any solution to Laplace's equation must have opposite signs. This is well-illustrated by one of the simplest solutions to Laplace's equation, the saddle surface, $f(x, y) = x^2 - y^2$.

D. The Minimal Surface Equation

The *minimal surface equation* provides a good model for the soap film surfaces created when any twisted wire loop is dipped into soap solution. In Cartesian coordinates, it is

$$\left(1 + f_y^2\right) f_{xx} - 2\, f_x\, f_y\, f_{xy} + \left(1 + f_x^2\right) f_{yy} = 0. \tag{30}$$

Solutions have the smallest possible surface area of any surface that fits across the wire loop (i.e. matches the boundary condition). But the left-hand side of this equation is the numerator of the standard expression for *mean curvature*. Equation (30) can also be written as

$$\kappa_p + \kappa_c\, S \left(1 + S^2\right) = 0. \tag{31}$$

E. An Idealized Steady-State Landscape Equation

An idealized mathematical model for steady-state fluvial landscapes has been proposed and analyzed by [4], [5], [6]. In coordinate-free form it is given by

$$\nabla \cdot \left(S^{\gamma - 1} \nabla f\right) = -R^{\star} \tag{32}$$

where $R^{\star} = R/q_1$ is a rescaled effective rainrate, γ is a parameter from a slope-discharge relationship of the form $q = q_1\, S^{\gamma}$ and q is unit-width discharge. It can be shown that this second-order, nonlinear PDE is *hyperbolic* when $\gamma < 0$, *parabolic* when $\gamma = 0$ and *elliptic* when $\gamma > 0$. The well-known empirical relations of hydraulic geometry imply that a mature, fluvial landscape will have $\gamma \approx -1$. By expressing (32) in Cartesian coordinates and then using (11) and (7), this PDE can be rewritten in the form

$$\kappa_p = (S/\gamma) \left(R^{\star}\, S^{-\gamma} - \kappa_c\right). \tag{33}$$

This allows us to make general statements about the character of solutions to (32) in terms of how plan and profile curvature must be related at each point. Note that S and $R^{\star}$ are always nonnegative. If we assume that $\gamma < 0$, we can deduce the following from (33).

- *Longitudinal profiles in valleys are always concave up.* Note that $\kappa_c < 0$ (valley) implies that $\kappa_p < 0$.
- *Narrower valleys have higher profile curvatures.* For a fixed S, $|\kappa_p|$ increases linearly with $|\kappa_c|$. Valley width can be defined as proportional to the radius of curvature or $w \propto r_c = 1/|\kappa_c|$.
- *Steeper valleys have higher profile curvatures.* For a fixed $\kappa_c < 0$ (valley), $|\kappa_p|$ is a rapidly increasing function of slope, S. In fact, if $\kappa_c = 0$ everywhere, as it is for a sheet-like surface with parallel contour lines, (33) can be solved as an ODE. For $\gamma = -1$ we find that $f(x, y) = (-1/R^{\star}) \ln(x + c)$.

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